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THE LAWS OF ELASTICO-VISCOUS FLOW

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When a solid is subjected to a strain beyond the 'Elastic Limit,'* its behavior may be summarized as follows:

First: The application of the stress results in a rapid elastic yield, which if inertia be negligible is practically instantaneous. If the stress be now removed, the specimen returns to its former position.†

Second: This is followed by a slower yielding whose rate, if the stress is not too great, diminishes with time, and which ultimately attains a constant value which may be zero.

If the stress be now removed the specimen returns almost instantaneously to a point short of its original position, and then continues at a much slower rate and ultimately comes to rest at a point short of its original position.

If the stress is too great, the slow yield may increase until rupture occurs.‡

The following may be considered as a provisional attempt to formulate the behavior of substances under stress by the simplest expressions which have been found to satisfy all the essential requirements.

* The term 'elastic limit' is very vague and should be replaced by limits which may be characterized as follows: a. *The first limit* is that within which the specimen returns instantly to its original zero. Beyond this limit, if stress be instantly removed, the specimen promptly returns to a position short of its original one, which may be designated the 'new zero.' b. *The second limit* is that beyond which the specimen does not return to its original position or to the 'new zero,' even after a long time. c. *The third limit* is that value of the stress which produces rapid yielding or rupture.

† In many cases the time interval between application and release of stress cannot be made sufficiently short for complete instantaneous recovery.

‡ Rupture may occur in consequence of such slow yielding, or, it may be practically instantaneous. In the former case, the result is due to separation of the viscous coupling; in the second to the snapping of the spring.

The formulae which follow are in fact sufficiently general to include every case thus far examined, including materials of widely different properties, such as lead, tin, copper, aluminum, zinc, iron, steel, quartz, glass, calcite, limestone, slate, marble, wax, pitch, gelatine and rubber. It may, however, be expected that a more thorough investigation will require modification in the formulae which may be made to fit special cases with greater accuracy.

The type of strain selected for this investigation is the torsion of cylindrical rods, as this is the only strain in which the form remains unaltered. It is very probable that the laws governing this special type may be made to include other distortions such as extension, compression, bending, etc.

Very decided changes may be expected from the effects of temperature and pressure,* but these may be taken into account by an appropriate alteration in the value of the 'constants' which enter into the formulae.

The apparatus employed for the investigation consisted in a light pulley with radius of 8 cm. over which passed two cords, the ends of which carried scale pans for holding weights.

The specimen to be investigated had a diameter of 12 mm. at the ends while the intervening portion (75 mm. long) had a diameter of 4 mm. One end was clamped to the supporting frame and the other to the pulley which rests on a knife-edge in the axis.

The tests consisted in measuring the angular position of the pulley by a micrometer at intervals of one minute while under a constant torque.

Laws of Elastico-viscous Flow.—The behavior of any solid under stress may be considered as the resultant of four elements:

a. The elastic displacement,

b. The elastico-viscous displacement,

c. The viscous displacement,

d. The lost motion.

These will be considered in turn.

The Elastic Displacement.—This is characterized by being approxi-

* A preliminary investigation of the effect of hydrostatic pressure on elasticity and on viscosity was begun several years ago, which it was hoped would show results in conformity with those which maintain in the body of the earth—whose enormous pressure produces an increase in both rigidity and viscosity sufficient to make the body of the earth (which at its actual temperature under ordinary conditions would certainly be in a molten state) as solid as steel. This expectation has been partially realized for a number of materials, metallic and non-metallic; the results notwithstanding certain anomalies—traceable to the effects of previous history—showing a perceptible increase in rigidity and a very marked increase in viscosity even with the relatively small pressures obtainable in the laboratory.

mately proportional to the stress and independent of time.* A closer approximation is given by

$$S_1 = C_1 P e^{h_1 P}$$

The Elastico-viscous Displacement.—This is manifested in a slow return when the stress is removed; and it is assumed that the same forces are brought into play during the direct motion.

This displacement is represented by the formula $S_2 = A_2(1 - e^{-\alpha \sqrt{t}})$ where $A_2 = C_2 P e^{h_2 P}$.

The Viscous Displacement.—Here the elastic force is absent or very small in comparison with the viscous resistance. The specimen does not return to zero even after a long time interval.† The viscous displacement is given by $S_3 = (Ft + F_0 t_0)^\rho - (F_0 t_0)^\rho$ in which $F = C_3 P e^{h_3 P}$ and F_0 the corresponding value, when P has the value P_0 during the time, t_0.

For a specimen which has not been subjected to previous strain the formula reduces to $S_3 = (Ft)^\rho$. Experiment gives $\rho = \frac{1}{2}$ approximately, until the specimen is near the rupture point when ρ approaches the value unity.

The Lost Motion.—If the stress be applied for a short time (even a small fraction of a second) the specimen does not return to the original zero. The difference between the original and the new zero is the lost motion, L.

It seems probable that the lost motion may be considered as a function of t such as t^r, where r is very small (less than 0.02 for zinc).

If this be considered as part of the viscous term

$$S_3 = A_3 f(t)$$

then the total viscous yield may be represented by

$$S_3 = A_3[f(t) + Ct^r]$$

(If the actual stress is between the limits 0 and P_0, $C = 0$)

The Return.—If after a time, t_0 the displacement has reached the value, S, and the stress is released, the specimen promptly returns to a displacement short of zero, and continues much more slowly in the same direction.

If the elastico-viscous displacement at the time, t_0 is given by

$$S_2 = A_2(1 - e^{-\alpha \sqrt{t_0}})$$

* Doubtless there is some viscous resistance to this displacement, but it is very small.

† In some cases it may be made to return to the original position by heating, or by alternation (alternate positive and negative diminishing stresses).

the corresponding return displacement at the time, t, counted from the instant of release, will be

$$R_2 = A_2 e^{-\alpha\sqrt{t}}(1 - e^{-\alpha\sqrt{t_0}})$$

To account for the viscous term assume*

$$F = \epsilon S^n \dot{S}$$

whence

$$S_3 = \left[\frac{1}{\rho\epsilon}\int F\,dt\right]^\rho, \qquad \rho = \frac{1}{n+1}$$

If F = constant, and F_0 the constant value of F during the preceding stress during the time, t_0,

$$S_3 = \frac{1}{\rho\epsilon}\left[(Ft + F_0t_0)^\rho - (F_0t_0)^\rho\right]$$

counting from the actual zero.

As shown by the formula, if the previous strain be considerable the new strain is relatively small. This strengthening by previous strain is one of the striking features of the behavior of every substance which exhibits viscous yield.

If, in this expression, F represents the actual stress, it assumes that the viscous force is proportional to the velocity, which is true for fluids; but for 'solid friction,' the force is independent of the velocity.

It may be assumed in the present case of internal viscosity of solids that the actual law may be between these two extremes, e.g.

$$P = a\,(\dot{S})^K \qquad \text{in which } K < 1$$

This would give P^n instead of P, or, in better agreement with experiment,

$$F' = PCe^{hP}$$

The elastico-viscous term is readily obtained by making the viscosity coefficient a function of the time.

Thus, if the restoring force be represented by $a\ s$ and the viscous resistance† by $\epsilon t^m\dot{S}$, the integration gives $S_2 = S_0(1 - e^{-\frac{\alpha}{r}t^r})$ where $\alpha = \frac{a}{\epsilon}$ and‡ $r = -m + 1$.

* Experiment gives $\rho = \frac{1}{2}$, (0.3 to 0.6) which makes $n = 1$. The usual assumption, $n = 0$ gives $\rho = 1$.

† The assumptions in both viscous and elastico-viscous hypotheses made the viscosity coefficient (that is the coefficient of $\dot{S}$) zero at the beginning of the motion and infinite at $t = \infty$ which is, of course, inadmissible. Instead of S^n and t^m we might substitute $(\beta + S^n)\ /\ (b + S^n)$ and $(\gamma + t^m)\ /\ (c + t^m)$ in which β/b and γ/c are very small; but the resulting equations are far less simple and are not appreciably more accurate in expressing the results than those here given.

‡ The usual assumption, $m = 0$, gives $r = 1$.

To determine the effect of temperature, the behavior of zinc, glass, ebonite, pitch and wax was studied. The results, together with the preceding may be summarized in the following formulae:*

$$S = A_1 + A_2 T_2 + A_3 T_3,$$

$$\begin{aligned} A_1 &= C_1 Pe^{h_1 P}, & C_1 &= E_0 + E_1 e^{K_1\theta}, \\ A_2 &= C_2 Pe^{h_2 P}, & C_2 &= E_2 e^{K_2\theta}, \\ A_3 &= C_3 Pe^{h_3 P}, & C_3 &= E_3 \theta (T - \theta)^{-1}, \\ T_2 &= 1 - e^{-\alpha\sqrt{t}}, & h &= b\,\theta, \end{aligned}$$

$$T_3 = \left(t + \frac{A_0}{A_3} t_0\right)^p - \left(\frac{A_0}{A_3} t_0\right)^p, \; p = p_0 + \frac{a}{1 + \left(\frac{\pi}{P\theta}\right)^m};$$

S = displacement (twist), P = applied torque,
t = time, t_0 = duration of previous stress,
θ = temperature, T = melting point,
h, k, α, E, P_0, a, b, m, π, constants.

* Instead of this series coupling, the following may be substituted: The unit consists of four elements: (1), (2) and (3) are in viscous contact with (4). (1) and (2) are in elastic coupling; and finally (3) of this unit is connected with (1) of the next following unit by an elastic coupling. The resulting formulae, however, are not essentially different from those here given.

A NEW EQUATION OF CONTINUITY

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This paper presents a pressure, volume, temperature relation which has been carefully compared with the greater part of the available experimental data during the past ten years. The equation, valid when only one type of molecule is present, is

$$p = \frac{R}{v - \delta} T - \frac{a}{(v - l)^2} \qquad (1)$$

where $\log \delta = \log \beta - \alpha/v$, and β, α, a, and b are constants; R is the universal gas constant.

The equation is based on considerations resulting from the inferences regarding atomic structure, obtained since the discovery of the negative electron. The model atom consists of a positive central portion about